CATEGORIFICATION OF THE SKEIN MODULE OF TANGLES

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Abstract. We generalize our previous work on categorification of Kauffman bracket skein module of surfaces, by extending our homology to tangles in cylinders over surfaces, $F \times [0,1]$. Our homology of 0-tangles and 1-tangles in D^3 coincides (up to normalization) with Khovanov link homology and the reduced Khovanov link homology.

We prove the basic properties of our homology. In particular, the short exact sequence of homologies of skein related tangles and the Künneth formula for the tensor product of tangles.

1. Introduction

In this paper we apply the ideas of [APS] to define "Khovanov like" homology of tangles in cylinders over surfaces.

A *marked surface* is an oriented compact surface, F, together with a distinguished finite set of "marked" points B in the boundary of F. *A framed tangle* in the cylinder over a marked surface (F, B) is a finite disjoint union of embedded bands, $b : [0,1] \times [0,1] \hookrightarrow F \times (-1,1)$, and of annuli, $a : S^1 \times [0,1] \hookrightarrow F \times (-1,1)$, such that for any band b, $b([0,1] \times [0,1]) \cap \partial F \times (-1,1) = b([0,1] \times \{0\}) \cup b([0,1] \times \{1\})$. Neither the bands nor annuli are oriented. However, we assume that all bands in a tangle have an integral framing. The midpoints of the arcs $b([0,1] \times \{0\})$, $b([0,1] \times \{0\})$ are called the endpoints of T and we assume that the set of endpoints of T coincides with the set of marked points of F. Each tangle T is represented by a tangle diagram composed of closed loops and arcs in F which represent the cores of annuli and of bands of T with blackboard framing. Therefore, each tangle diagram is a compact 1-manifold D properly embedded into F and such that $D \cap \partial F = B$. Since tangles are considered up to an ambient isotopy fixing

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their endpoints, two tangle diagrams are equivalent if they differ by second, third, and balanced first Reidemeister moves.

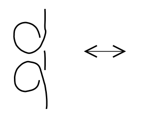


Fig. 1: Balanced first Reidemeister move

We denote the set of all tangles in the cylinder over marked surface (F, B) by $\mathcal{T}(F, B)$. Additionally, we denote the set of all diagrams of tangles in $\mathcal{T}(F, B)$ with no crossings and no contractible closed loops by $\mathcal{B}(F, B)$. Since different diagrams in $\mathcal{B}(F, B)$ represent non-isomorphic tangles, $\mathcal{B}(F, B) \subset \mathcal{T}(F, B)$. Note that $\mathcal{B}(F, \emptyset)$ is the set of all finite collections of non-intersecting nontrivial simple closed loops in F considered up to homotopy (including the empty collection). Observe also that if $B \subset \partial D^2$ is a set of $2n$ points then $\mathcal{B}(D^2, B)$ has the n-th Catalan number of elements, $\frac{1}{n+1}\binom{2n}{n}$.

For any tangle T in $\mathcal{T}(F, B)$ we define homology groups $H_{i,j,s}(T)$, for $i, j \in \mathbb{Z}$ and $s \in \mathcal{B}(F, B)$, which are invariant under isotopies of T and which generalize the homology groups of [APS]. Its properties are discussed in Sections 3 and 4.

2. Definition of Homology

Let $\mathcal{B}'(F, B) \subset \mathcal{B}(F, B)$ be the set of all tangle diagrams in F with no closed loops. Consequently, diagrams in $\mathcal{B}'(F, B)$ are composed of non-intersecting arcs only. Let $\mathcal{C}(F)$ be the set of all unoriented non-contractible simple closed curves in F considered up to homotopy. By separating arcs from closed loops in a tangle, we embed $\mathcal{B}(F, B)$ into $\mathbb{N}\mathcal{C}(F) \times \mathcal{B}'(F, B)$, where $\mathbb{N} = \{0, 1, 2, ...\}$. We are about to define homology groups of tangles T in $\mathcal{T}(F, B)$, $H_{i,j,s,b}(T)$, indexed by $i, j \in \mathbb{Z}$, $s \in \mathbb{Z}\mathcal{C}(F)$ and $b \in \mathcal{B}'(F, B)$. The following proposition is proved at the end of Section 3:

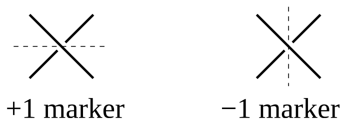
Proposition 1. *(1) $H_{i,j,s,b}(T)$ is isomorphic to $H_{i,j,|s|,b}(T)$, where $|s| = \sum_\gamma |s_\gamma| \cdot \gamma$ for $s = \sum_\gamma s_\gamma \cdot \gamma$ in $\mathbb{Z}\mathcal{C}(F)$.*
(2) $H_{i,j,|s|,b}(T) = 0$ if $(|s|, b) \in \mathbb{N}\mathcal{C}(F) \times \mathcal{B}'(F, B)$ does not lie in the image of $\mathcal{B}(F, B)$.

Corollary 2. *Alternatively, our homology groups can be indexed by $i, j \in \mathbb{Z}$ and $s \in \mathcal{B}(F, B)$ (as promised in the introduction).*

Nonetheless, indexing by $s \in \mathbb{Z}\mathcal{C}(F)$ and $b \in \mathcal{B}'(F, B)$ is useful for technical reasons.

Consider a tangle diagram D in F whose crossings are ordered by consecutive integers $1, 2, ...$ Following [Vi1, APS], a *state* S of D is an assignment of $+$ or $-$ sign to each of the crossings of D and an additional assignment of

$+$ or $-$ sign to each closed loop in the diagram D_S obtained by smoothing the crossings of D according to the following convention:



(Note that arcs of D_S do not carry any labels.) Our construction of the chain complex associated with D is analogous to that of [APS]:

For any state S of D let

$$\begin{aligned} I(S) &= \sharp\{positive\ crossing\ markers\} - \sharp\{negative\ crossing\ markers\}, \\ J(S) &= I(S) + 2\sharp\{positive\ contr.\ circles\} - 2\sharp\{negative\ contr.\ circles\}, \end{aligned}$$

where "contr" stands for "contractible".

Let $\Phi(S)$ be an element of $\mathcal{B}(F,B)$ obtained from D_S by removing all closed loops. Denote the non-contractible loops in D_S by $\gamma_1,...,\gamma_n$. If these loops are marked by $\varepsilon_1,...,\varepsilon_n \in \{+1,-1\}$, then let

$$\Psi(S) = \sum_i \varepsilon_i \gamma_i \in \mathbb{Z}\mathcal{C}(F),$$

cf. [APS].

Let $\mathcal{S}_{i,j,s,b}(D)$ be the set of all states S of D with $I(S) = i, J(S) = j$, $\Psi(S) = s$, and $\Phi(S) = b$. Let $\mathcal{C}_{i,j,s,b}(D)$ be the free abelian group generated by states in $\mathcal{S}_{i,j,s,b}(D)$. We define the incidence number between states following [APS, Definition 3.1]:
$[S : S']_v = 1$ if the following four conditions are satisfied:

(1) the crossing v is marked by $+$ in S and by $-$ in S',
(2) S and S' assign the same markers to all the other crossings,
(3) the labels of the common circles in S and S' are unchanged,
(4) $J(S) = J(S')$, $\Psi(S) = \Psi(S')$, $\Phi(S) = \Phi(S')$.

Otherwise $[S : S']_v$ is equal to 0.

The types of incident states at a crossing v which involve loops only are listed in [APS, Table 2.1]. The only two other types of incident states are as follows:

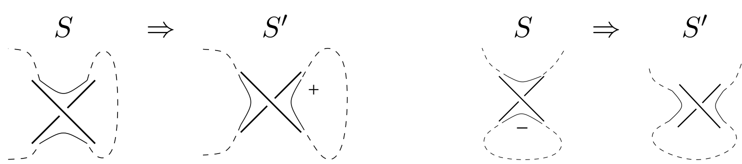


Fig 2. Incident states involving arcs

The labeled loops appearing in this diagram are contractible.

Let $t(S,v)$ denote the number of negative markers assigned to crossings bigger than v. Then

$$d_{i,j,s,b} : C_{i,j,s,b}(D) \to C_{i-2,j,s,b}(D)$$

is defined by

$$d_{i,j,s,b}(S) = \sum_v (-1)^{t(S,v)} d_{i,j,s,b,v}(S),$$

where

$$d_{i,j,s,b,v}(S) = \sum [S : S']_v S',$$

and the sum is over all states of D. Clearly, it is enough to consider states in $\mathcal{S}_{i-2,j,s,b}(D)$ only. As in [APS], d has degree -2 with respect to the first index. Note also that $C_{i,*,*,*}(D) = 0$ for i not equal to the number of crossings of D mod 2.

Remark 3. *For any link diagram D in a surface F, $(\mathcal{S}_{*,*,*,\emptyset}(D), d)$ is the chain complex introduced in* [APS].

In the next section we will prove:

Proposition 4. *For any marked surface (F, B) and $T \in \mathcal{T}(F, B)$, $d^2 = 0$.*

Therefore for any abelian group G, $(C_{*,j,s,b}(D) \otimes G, d_{*,j,s,b})$ is a chain complex whose homology groups we denote by $H_{*,j,s,b}(D; G)$, as usual abbreviating $H_{i,j,s,b}(D; \mathbb{Z})$ to $H_{i,j,s,b}(D)$. By the argument of [APS, Sec. 10.1], $H_{i,j,s,b}(D; G)$ does not depend (up to isomorphism) on the ordering of crossings of D.

3. Embeddings into surfaces, Reidemeister moves

Let (F, B) and (F', B') be marked surfaces with equal numbers of boundary components, such that there is a preserving orientation homeomorphism $\phi : \partial F \to \partial F'$ mapping B onto B'. Then for any $b \in \mathcal{B}'(F, B)$ and $b' \in \mathcal{B}'(F', B')$, $b \cup_\phi b'$ is a disjoint union of simple closed curves in the closed surface $F \cup_\phi F'$. We leave the proof of the following easy statement to the reader.

Lemma 5. *For any marked surface (F, B) and any $b \in \mathcal{B}'(F, B)$ there is a marked surface (F', B'), $b' \in \mathcal{B}'(F', B')$ and a homeomorphism $\phi : \partial F \to \partial F'$ as above such that the components of $b \cup_\phi b'$ are not homotopic to any loops in F and are not homotopic one to another.*

Let (F, B), (F', B'), $b \in \mathcal{B}'(F, B)$, $b' \in \mathcal{B}'(F, B)$ satisfy the assumptions of Lemma 5 and let c be a state of $b \cup_\phi b'$. For any $T \in \mathcal{T}(F, B)$ and its state $S \in \mathcal{S}_{i,j,s,b}(T)$, let $\Lambda(S) \in \mathcal{S}_{i,j,s+c,\emptyset}(T \cup_\phi b')$ be the state $S \cup_\phi b'$ obtained by labeling the loops of $b \cup_\phi b'$ as in c.

Lemma 6. *Λ extends to an isomorphism*

$$\Lambda : \mathcal{C}_{i,j,s,b}(T) \to \mathcal{C}_{i,j,s+c,\emptyset}(T \cup_\phi b')$$

commuting with the differentials.

Proof. Λ is $1-1$ since no loops in $b \cup_\phi b'$ are homotopic to each other nor are homotopic to loops in s. Obviously, Λ is onto as well. Since states $S, S' \in \mathcal{C}_{*,j,s,b}(T)$ are incident if and only if $S \cup_\phi b'$ and $S' \cup_\phi b'$ are incident, the statement follows. □

Since by Remark 3, $\mathcal{C}_{*,j,s+c,\emptyset}(T \cup_\phi b')$ is a chain complex, Lemma 6 implies Proposition 4 and, together with (7) of [APS], it implies Proposition 1.

Now, Lemma 6 and [APS, Theorem 2] imply

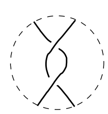
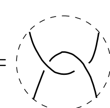
Theorem 7. *Let $i, j \in \mathbb{Z}$, $s \in \mathbb{Z}\mathcal{C}(F)$, D be a diagram of a tangle in $\mathcal{T}(F, B)$, $b \in \mathcal{B}'(F, B)$,*
(1) If D' is related to D by the first Reidemeister move consisting of adding a negative kink to D then $H_{i,j,s,b}(D') = H_{i-1,j-3,s,b}(D)$.
(2) $H_{i,j,s,b}(D)$ is invariant (up to an isomorphism) under the second and third Reidemeister moves.

Additionally, by Lemma 6 for any diagram D of a tangle in $\mathcal{T}(F, B)$ the groups $H_{i,j,s,b}(D)$ coincide with stratified homology groups $H_{i,j,s+c}(\bar{D})$ defined in [APS] for the link diagram $\bar{D} = D \cup_\phi b'$ constructed above.

Further properties of our homology groups are described below. From now on we index our homology groups by $i, j \in \mathbb{Z}$ and $b \in \mathcal{B}(F, B)$, as proposed in Corollary 2.

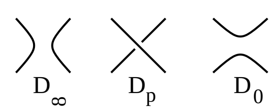
4. Properties

4.1. **Twisting the endpoints.** For any tangle $T \in \mathcal{T}(F, B)$ and any component $C \subset \partial F$ let $\lambda_C(T)$ denote a tangle obtained from T by shifting its endpoints in C counterclockwise by one:

$T =$  $\lambda(T) =$ 

Note that $\lambda^n(T) = T$ for $T \in \mathcal{T}(F, B)$ if $|B| = n$ and $F = D^2$, but not necessarily for other surfaces. In any case, we have $H_{i,j,s,b}(T) = H_{i,j,s,\lambda_c(b)}(\lambda_c(T))$.

4.2. **Short exact sequence.** Any three skein related tangle diagrams in F



define a short exact sequence

$$(1) \qquad 0 \to C_{*jsb}(D_\infty) \xrightarrow{\alpha} C_{*,j-1,s,b}(D_p) \xrightarrow{\beta} C_{*,j-2,s,b}(D_0) \to 0$$

where the maps $\alpha : C_{i,j,s,b}(D_\infty) \to C_{i-1,j-1,s,b}(D_p)$ and $\beta : C_{i,j-1,s,b}(D_p) \to C_{i-1,j-1,s,b}(D_0)$ are defined as in [APS, Sec 7]. This sequence leads to the long exact sequence

(2)

$$... \to H_{i,j,s,b}(D_\infty) \xrightarrow{\alpha_*} H_{i-1,j-1,s,b}(D_p) \xrightarrow{\beta_*} H_{i-2,j-2,s,b}(D_0) \xrightarrow{\partial} H_{i-2,j,s,b}(D_\infty) \to ...$$

4.3. **Categorification of the skein modules of tangles.** Formal $\mathbb{Z}[A^{\pm 1}]$-linear combinations tangles in $\mathcal{T}(F,B)$ quotiented by the skein relations of the Kauffman bracket

$$\diagup\!\!\!\!\diagdown = A \asymp + A^{-1})(, \quad L \cup \bigcirc = -(A^2 + A^{-2})L,$$

form the relative Kauffman bracket skein module $\mathcal{S}(F,B)$, cf. [Pr]. The set $\mathcal{B}(F,B)$ forms a natural basis of $\mathcal{S}(F,B)$. For any tangle $T \in \mathcal{T}(F,B)$, we denote its coordinates in this basis by $\langle T\rangle_b$, $b \in \mathcal{B}(F,B)$. Hence

$$T = \sum_{b \in \mathcal{B}(F,B)} \langle T\rangle_b b \in \mathcal{S}(F,B).$$

Let $\chi_A(H_{**})$ denote the polynomial Euler characteristic of a bigraded group H_{**} defined as in [APS, Sec 1]:

$$\chi_A(H_{**}) = \sum_{i,j} A^j (-1)^{\frac{j-i}{2}} rk H_{i,j}. \tag{3}$$

In the following proposition we index our tangle homology groups by $i, j \in \mathbb{Z}$ and $b \in \mathcal{B}(F,B)$ as in Corollary 2:

Proposition 8. *For any* $T \in \mathcal{T}(F,B)$ *and* $b \in \mathcal{B}(F,B)$, $\chi_A(H_{**b}(T)) = \langle T\rangle_b$

Proof. By definition

$$\left\langle \diagup\!\!\!\!\diagdown \right\rangle_b = A \left\langle \asymp \right\rangle_b + A^{-1} \left\langle)(\right\rangle_b, \quad \langle L \cup \bigcirc\rangle_b = -(A^2 + A^{-2}) \langle L\rangle_b ,$$

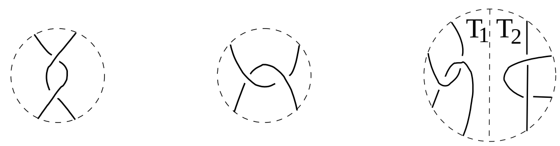
and, by (2), analogous identifies hold for $\chi_A(H_{**b}(T))$. Therefore, it is enough to assume that the tangle diagram T has no crossings and no trivial components. Under these assumptions $T \in \mathcal{B}'(F,B)$ and

$$\chi_A(H_{**b}(T)) = \chi_A(C_{**b}(T)) = rk\, C_{0,0,b}(T) = \delta_{b,T} = \langle T\rangle_b,$$

where $\delta_{b,T} = \begin{cases} 1 & \text{if } b = T \\ 0 & \text{otherwise.} \end{cases}$ □

4.4. **Reduced Link Homology.** For any link diagram L with a specified one of its components, Khovanov defined its *reduced homology*, $\tilde{H}_{i,j}(L)$, [K3, Sh]. His construction has the following interpretation in our setting: If L' is a 1-tangle obtained by cutting L at an arbitrary point of its specified component and its endpoints are p_1 and p_2, then $\tilde{H}_{i,j}(L)$ is isomorphic (up to normalization of indices) to $H_{i,j,\alpha}(L')$, where α is the unique element of $\mathcal{B}(D^2, \{p_1, p_2\})$. Since different cutting points on a given component of a link give isomorphic 1-tangles, we get another proof that $\tilde{H}_{i,j}(L)$ does not depend on the choice of the cutting point on the distinguished component L.

4.5. **Tensor product of tangles.** Consider a decomposition of a disk D^2 into two disks D_1, D_2 by a properly embedded arc whose endpoints are disjoint from a finite set $B \subset \partial D^2$. Let $B_1 = B \cap \partial D_1$ and $B_2 = B \cap \partial D_2$. In this situation, a union of tangles $T_i \in \mathcal{T}(D_i, B_i)$, $i = 1, 2$, is a tangle in $\mathcal{T}(D^2, B)$ which we denote by $T_1 \otimes T_2$.



T_1 T_2 $T_1 \otimes T_2$

Fig. 3: A tensor product of tangles.

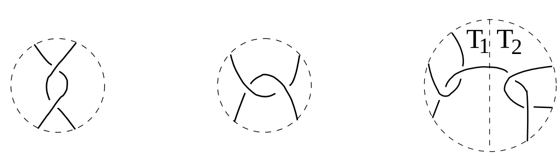
Note that $T_1 \otimes T_2$ is not well defined for tangles T_1, T_2 in disjoint disks, since several ways of gluing of these disks are possible. This issue is irrelevant however in the calculation of homology of $T_1 \otimes T_2$. To see that, observe first that if $T_i \in \mathcal{T}(D_i, B_i)$, $i = 1, 2$, then $H_{**s}(T_1 \otimes T_2) = 0$ unless $s = s_1 \otimes s_2$ for some $s_1 \in \mathcal{B}(D_1, B_1)$, $s_2 \in \mathcal{B}(D_2, B_2)$. If $s = s_1 \otimes s_2$ then s_1 and s_2 are unique and $C_{**s}(T_1 \otimes T_2)$ is the tensor product of (filtered) chain complexes, $C_{**s_1}(T_1)$ and $C_{**s_2}(T_2)$. By the Künneth formula we have a short exact sequence

$$0 \to \bigoplus_{\substack{i_1+i_2=i,\\ j_1+j_2=j}} H_{i_1,j_1,s_1}(T_1) \otimes H_{i_2,j_2,s_2}(T_2) \to H_{i,j,s}(T_1 \otimes T_2) \to$$

$$\to \bigoplus_{\substack{i_1+i_2=i-1,\\ j_1+j_2=j}} Tor(H_{i_1,j_1,s_1}(T_1), H_{i_2,j_2,s_2}(T_2)) \to 0 \tag{4}$$

which is non-canonically split. In particular the homology groups of $T_1 \otimes T_2$ are determined by the homology groups of T_1 and T_2.

4.6. **Reduced tensor products of tangles.** Again, consider a decomposition of a disk D^2 into two disks D_1, D_2 by a properly embedded arc whose endpoints are disjoint from a finite set $B \subset \partial D^2$. Let p be a selected point inside that arc and let $B_1 = B \cap \partial D_1 \cup \{p\}$ and $B_2 = B \cap \partial D_2 \cup \{p\}$. In this situation, union of tangles $T_i \in \mathcal{T}(D_i, B_i)$, $i = 1, 2$, is a tangle in $\mathcal{T}(D^2, B)$ which we call a reduced tensor product of T_1 and T_2 and denote by $T_1 \otimes_1 T_2$.



T_1 T_2 $T_1 \otimes_1 T_2$

Fig. 4: A reduced tensor product of tangles

As for tensor products, the homology groups of reduced tensor products $T_1 \otimes_1 T_2$ are related to homology groups of T_1 and T_2 by Künneth formula. As before, if $H_{**s}(T_1 \otimes_1 T_2) \neq 0$ then $s = s_1 \otimes_1 s_2$ for unique $s_i \in \mathcal{B}(D_i, B_i)$, $i = 1, 2$. In this situation $C_{**s}(T_1 \otimes_1 T_2)$ is the tensor product of (filtered) chain complexes $C_{**s_1}(T_1)$ and $C_{**s_2}(T_2)$ inducing a non-canonically split short exact sequence

$$0 \to \bigoplus_{\substack{i_1+i_2=i,\\ j_1+j_2=j}} H_{i_1,j_1,s_1}(T_1) \otimes H_{i_2,j_2,s_2}(T_2) \to H_{i,j,s}(T_1 \otimes_1 T_2) \to$$

$$\to \bigoplus_{\substack{i_1+i_2=i-1,\\ j_1+j_2=j}} Tor(H_{i_1,j_1,s_1}(T_1), H_{i_2,j_2,s_2}(T_2)) \to 0 \tag{5}$$

4.7. **Open questions and final comments.** If T is union of tangles T_1, T_2 with more than a single pair of their endpoints identified then the dependence of homologies of T_1, T_2 and T is far more complicated. In the simplest case, when T_2 is a 1-tangle composed of a single arc whose endpoints are identified with two consecutive endpoints of T_1, one can construct a spectral sequence converging to $H_{***}(T)$ whose E^2 term is composed of homology groups of T_1.

Given a tangle T in $[0,1] \times [0,1]$ with specified top $2n$ and bottom $2m$ points, Khovanov constructed homology groups $\mathcal{H}_{i,j}(T)$, which are invariant under isotopies of T. Additionally each such group is an (H^n, H^m)-bimodule for certain rings H^n, H^m defined in [K2, Ja]. As groups, $\mathcal{H}_{i,j}(T)$ are sums of Khovanov homology groups $H_{i,j}(L)$ over all links L which are plat closures of T. (A plat closure of T is a link obtained by closing its $2n$ top points by n non-intersecting arcs and, analogously, its $2m$ bottom points by m non-intersecting arcs.) We expect that for any tangle T there is a spectral sequence converging to $\mathcal{H}_{*j}(T)$ whose summands of E^2 are sums of certain groups $H_{ijs}(T)$. We do not know if there is a more explicit relation between Khovanov's and our homology groups. Nor we can interpret the bimodule structure of Khovanov's groups in our setting.

There is a parallel theory of tangle homologies due to D. Bar-Natan. At present time we do not know the relation between his and our tangle homologies.

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